STATEMENT OF TEACHING

Introduction

Mathematics is a collaborative process and should be available to everyone. I think it would be difficult to find a mathematician who disagrees with this basic premise. And yet, we don't live up to it. Very little mathematical writing is accessible to lower level undergraduate math students, let alone the general public. Talks by mathematicians often seem geared toward explaining the subject matter to two or three specialists in the room, with little regard for the rest of the audience. Why is it like this?

I don't pretend to have an answer, much less a solution; I am inclined to think that it is less a moral failing and more that it is a legitimately difficult problem. Even so, I am very interested in how we communicate mathematics and who we communicate it to. I suppose a significant part of my overall mission has been to make mathematics more generally accessible. Below are some of my thoughts on how to do this.

Mathematical Visualization and Interactability

Consider the following two illustrating examples:

(1) Show that there exists a function $z = f(x, y)$ such that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, but the limits $\lim_{x\to 0} \lim_{y\to 0} f(x, y)$, $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ do exist and are equal.

(2) Let ΔABC be a triangle on the sphere. Prove that $\angle A + \angle B + \angle C - \pi$ is equal to the area of ΔABC.

The first is a common type of problem for students of multivariable calculus; the second is Girard's theorem. Both have a single, unifying characteristic: they have simple solutions that are easy to understand if you can just see a decent picture of what is going on. For the first problem, I offer the function $f(x, y) = \frac{xy}{x^2+y^2}$ and the illustration in Figure 1. For the second, I offer Figure 2 as a guide to the proof. Both are my own work.

I have devoted a lot of time to seeing how we can make mathematics both visual and interactable, because it is one of the easiest ways to build intuition. Modern graphical software is incredible. Most obviously, it can produce highly detailed drawings that would have been impossible on a blackboard, but it also allows the creation of mathematical models that can be rendered and manipulated in real time, either in the classroom or at home. I have found this to be useful for teaching calculus students (who often lack training in dealing with abstract problems), but it is helpful at all levels. As an example, one can find a Mathematica notebook on my website that I built for teaching basic cryptography.

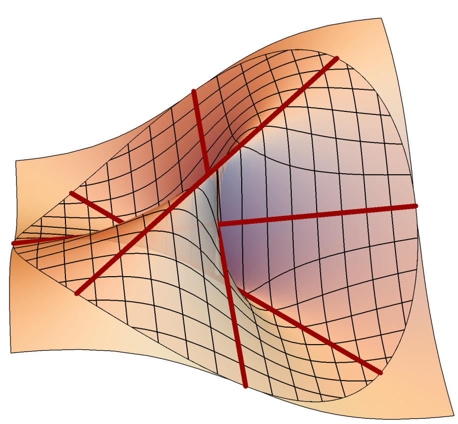

Figure 1. Graph of $z = \dfrac{xy}{x^2 + y^2}$

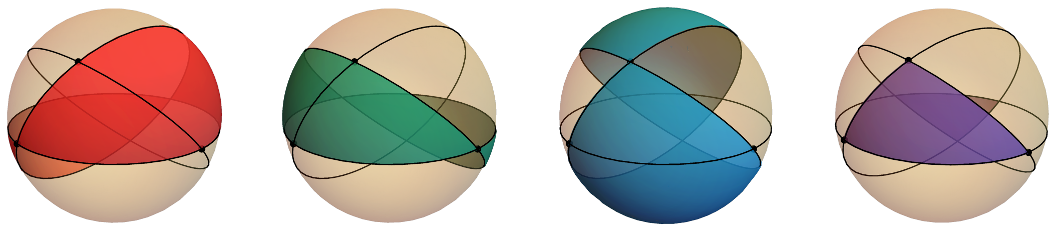

Figure 2. A hint of the proof of Girard's theorem.

It is certainly not the case that we can illustrate everything, but there is so much that we can, and I think we should try to do more. This is the philosophy with which I wrote *Linear Fractional Transformations: An Illustrated Introduction*, to be published in Springer's *Undergraduate Texts in Mathematics* sometime next year—there are quite literally hundreds of diagrams in that book. (An excerpt from the preprint of this book is available on my website.) It is the same philosophy that I hope to bring to future books and to the classroom, as well.

Mathematical Applications

My undergraduate degree was in mathematics and physics—this allows me to talk about why a subject like calculus matters to physicists. However, few of my students are either mathematicians or physicists, and I found myself frustrated by the fact that I was lacking in concrete applications that might be of interest to biologists, chemists, computer scientists, etc.. To counteract this, I have tried to read about the uses of mathematics in other subjects, and eventually, it pushed me to start the Mathematical Applications project[1]. The idea was to create something akin to a blog where vetted authors could write about how pure mathematics is used in other disciplines. We have done reasonably well—we now have over 50,000 subscribers, and many descriptions of applications for everything ranging from plane geometry to algebraic geometry. I have learned a great deal from this endeavor.

Adapting this to the classroom can be done in many ways. The most direct approach is to simply spend some time during lecture talking about possible applications. Many subjects are naturally well-suited to this. I was invited to give a talk in August of 2019 to the high school students at the Massachusetts Academy of Math and Science—ostensibly, this was a talk about cryptography, but in many ways, it was an introduction to working with modular arithmetic. In my calculus courses, I always take some time after introducing derivatives and then integrals to list possible ways these abstract mathematical objects can be implemented. This works even better if you can have your students make their own suggestions.

It is also possible to be more hands-on, which I try to do as well. My favorite example thus far is for teaching related rates—I bring in a pole to class, and lean it against the wall. We set our goal: to come up with a model that would describe with what speed the pole would hit the floor when it inevitably slid down the wall. We start with the assumptions that the end of the pole resting against the wall will continue to rest against it the entire way and that the end sliding across the floor will not slow down. Working through the problem either in groups or on the board, one comes up with a surprising conclusion: the pole will crash into the floor with infinite speed!

I give my class some time to contemplate what went wrong. Typically, after a minute they come up with two suggestions: first, that perhaps the pole *would* slow down (perhaps due to friction); and secondly, perhaps the pole would not continue resting against the wall the entire time. We then carry out the experiment, and precisely one of these two possibilities occurs, depending on the weight of the pole and stickiness of the floor. I find that this is a good opportunity not just to review related rates or limits, but to demonstrate more abstract principles: that even simple mathematical models can make important predictions; that wrong answers can be useful; and, most importantly, that calculus tangibly works.

[1]Hosted on Quora: `https://mathematicalapplications.quora.com`

The Socratic Method

In my experience, the answers that students come up with themselves are the ones that stay with them the longest. I had the privilege of working for PROMYS as a counselor in the summer of 2010—we had the core rule that a counselor should answer a student's question with another question whenever possible. The philosophy was that the interaction between instructor and student should be Socratic: it is the instructor's duty to frame and guide the discussion so that the student arrives to the conclusion on their own.

I am a fan of this approach. It is possible to make exceptional visualizations, to come up with clear applications of the material, and not to have students retain any of it because they did not get an opportunity to try it for themselves. In all of my classes, I regularly have students work together in groups. It gives them very necessary exercise; it gives me opportunity to talk with students individually and to evaluate where they are struggling. By working with smaller groups of students, I can talk less and listen more.

I have found this approach to be incredibly effective and rewarding outside of the traditional classroom setting as well. I worked as a graduate mentor at the DIMAC's REU program in the summer of 2015, and at SUMRY in the summers of 2016 and 2017, guiding a small group of students in performing mathematical research. In 2016, for example, I co-mentored three students working on determining how often do imaginary quadratic fields embed into rational quaternion algebras. The first week was a flurry of lectures as I gave them a crash course in quaternion algebras, p-adic analysis, and quadratic forms. However, it was in the second week, when I was away at a conference and could only communicate with them via Skype, that the true magic happened: they took everything that I had taught them and started putting it together, producing new results. They consulted with me, but the project had become their own. Seeing how excited they were to be working independently was one of my proudest teaching moments.

The project in 2017 was perhaps even more successful—I guided three students in a rather open-ended project related to Ulam sequences. There were dozens of open questions that could have been asked, and this gave each of them the freedom of working on something that they were especially passionate about, but which still tied together into the bigger picture. The results were marvelous: we discovered a previously unknown rigidity phenomenon for families of Ulam sequences, which culminated in two co-authored papers—one published in the Journal of Number Theory, and the other in the undergraduate research journal Involve. One of those students is now writing his own papers in graduate school.

Accessibility

One problem that is near and dear to my heart is how to make mathematics more accessible. There are at least two senses in which I mean this: first, that there should be more publicly available resources for mathematics; second, that mathematical literature and talks should be easier to follow for a larger audience where possible. Since 2015, I have been an active user of the Q-and-A site Quora, where I have regularly written on mathematics. It is admittedly a somewhat unusual choice—I think that more mathematicians tend to choose sites like Stackexchange or their own personal blogs for such things. While I think such websites are also important, they have the serious drawback that they largely only reach people who are already invested in mathematics. The Quora audience is much more diverse. One of my proudest achievements on that site was a post on how sample and population size affects polling, which has been viewed over 100k times. This is a rare opportunity to spread statistical literacy, which is sorely needed.

Breaking down advanced material to a more manageable level is certainly vital when communicating to laypersons, but that same approach is useful everywhere. One of the biggest challenges of writing *Linear Fractional Transformations*, for example, was taking theorems that are often proved using complex analysis or Lie groups, and finding creative ways to remove those requirements. I also had to find how to introduce some of the basics of Ulam sequences and what is known at a high school level—for my sister, specifically. She is 16 years my junior, and did a project for a science fair on this. I am hoping that we will be able to publish some of her findings in an undergraduate journal at some point in the near future.