How to Keep a Secret

Arseniy (Senia) Sheydvasser

August 27, 2019

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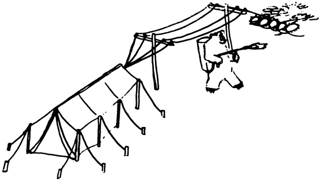

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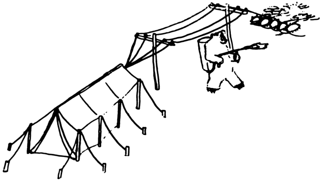
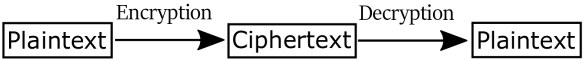


- Better solution:



Ancient Cryptography:

> If he had anything confidential to say, he wrote it in cipher, that is, by so changing the order of the letters of the alphabet, that not a word could be made out. If anyone wishes to decipher these, and get at their meaning, he must substitute the fourth letter of the alphabet, namely D, for A, and so with the others.
>
> Suetonius, Life of Julius Caesar

A	B	C	D	E	F	G	H	I	K	L	M	N	O	P	Q	R	S	T	V	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
20	21	22	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
X	Y	Z	A	B	C	D	E	F	G	H	I	K	L	M	N	O	P	Q	R	S	T	V

Caesar Cipher:

Example

E	G	O	I	N	E	V	I	T	A	B	I	L	I	S
4	6	13	8	12	4	19	8	18	0	1	8	10	8	17
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
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  - Encryption: $x \mapsto x + k \mod 23$ for some value k (called the *key*).
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Example

Take $a_1 = 5$, $a_2 = 21$, $b = 8$.

A	V	E	N	G	E	R	S	A	S	S	E	M	B	L	E
0	21	4	13	6	4	17	18	0	18	18	4	12	1	11	4
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
8	9	2	21	12	2	15	20	8	20	20	2	16	13	11	2
I	J	C	V	M	C	P	U	I	U	U	C	Q	N	L	C

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- MORAL: Any good encryption scheme should obfuscate any statistical properties of the original message.

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- It should be that even if an attacker learns how your system works currently, it should be easy to swap out some basic things and maintain security.

Basic Terminology:

$\boxed{\begin{array}{c}\text{Plaintext}\\ m\end{array}}$ $\mathcal{M}$

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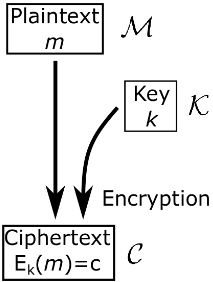

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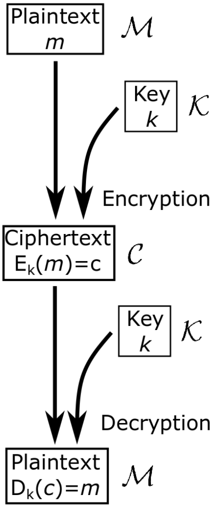
- Decryption algorithm D_k
- Require $D_k(E_k(m)) = m$.



Perfect Secrecy:

Definition

We say that $(\mathcal{M}, \mathcal{K}, \mathcal{C}, E_k, D_k)$ is *information-theoretically secure* if for any ciphertext c, the probability that it decodes to a message m is independent of m.

$$\Pr_{k \in \mathcal{K}}\left(E_k(m_1) = c\right) = \Pr_{k \in \mathcal{K}}\left(E_k(m_2) = c\right).$$

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Question

Do there exist any information-theoretically secure cryptosystems?

One-Time Pad (One Bit Version):

- Suppose you only have one of two possible messages, for example:
  - 0 = Don't attack
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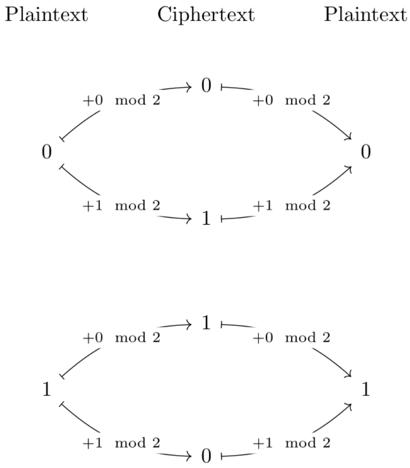

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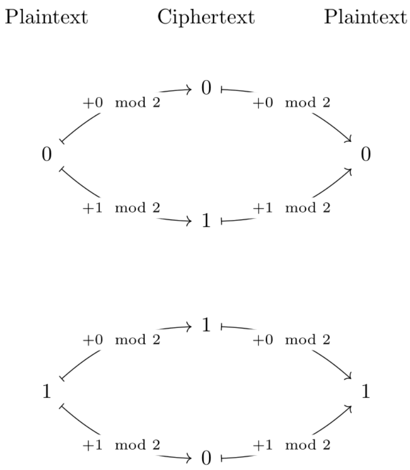

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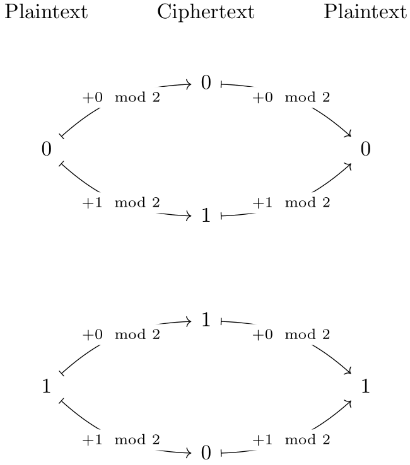

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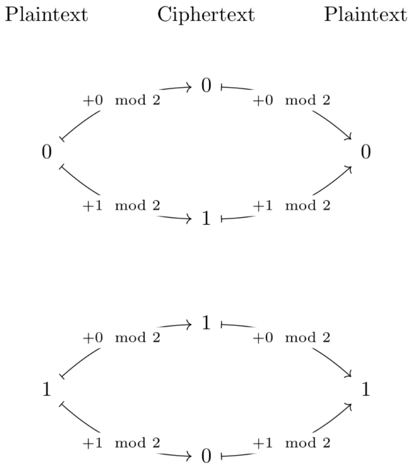
- This is information-theoretically secure!



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- *Exercise:* Prove that the one-time pad is information-theoretically secure.

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- *Step 5*: Remove padding, undo encoding.

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Cons:

- Fixed length of message.
- Key is as long as the message.
- Vulnerable to known-plaintext attacks.
- Keys cannot be reused without losing secrecy.

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- In practice, we require a less stringent security requirement: we only ask that it be *computationally difficult* to find statistical differences between ciphertexts and randomly generated strings.
- Even so, the essential backbone of the one-time pad is still used.

High-level description of the algorithm [edit]

1. `KeyExpansion`—round keys are derived from the cipher key using Rijndael's key schedule. AES requires a separate 128-bit round key block for each round plus one more.
2. Initial round key addition:
    1. `AddRoundKey`—each byte of the state is combined with a block of the round key using bitwise xor.
3. 9, 11 or 13 rounds:
    1. `SubBytes`—a non-linear substitution step where each byte is replaced with another according to a lookup table.
    2. `ShiftRows`—a transposition step where the last three rows of the state are shifted cyclically a certain number of steps.
    3. `MixColumns`—a linear mixing operation which operates on the columns of the state, combining the four bytes in each column.
    4. `AddRoundKey`
4. Final round (making 10, 12 or 14 rounds in total):
    1. `SubBytes`
    2. `ShiftRows`
    3. `AddRoundKey`