Applications of Orders with Involution

Arseniy (Senia) Sheydvasser

March 2, 2019

(Searching for) Applications of Orders with Involution

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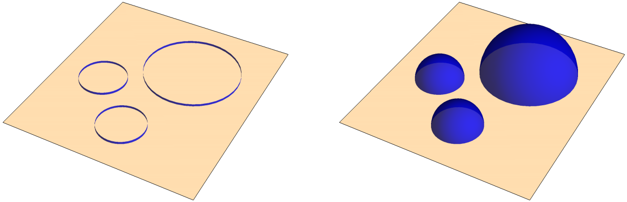

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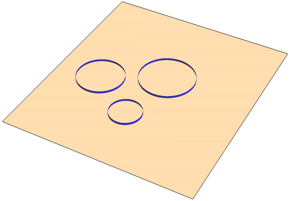

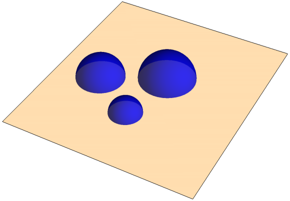

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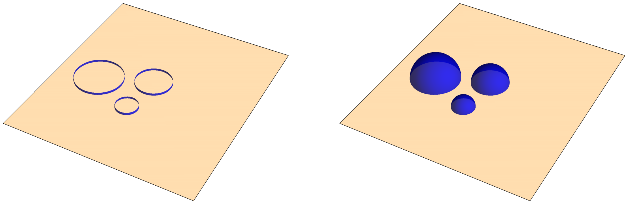

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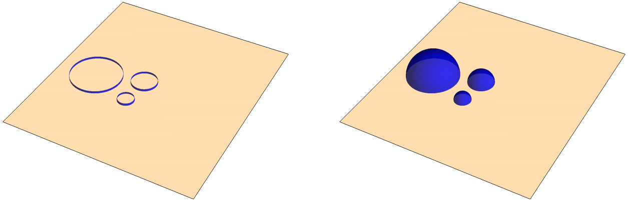

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Vahlen's Matrices

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$$SL^{\ddagger}(2, H_{\mathbb{R}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, H_{\mathbb{R}}) \middle| ab^{\ddagger}, cd^{\ddagger} \in H_{\mathbb{R}}^{+}, ad^{\ddagger} - bc^{\ddagger} = 1 \right\}$$

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Action on $\mathbb{R}^3 \cup \{\infty\} = H_\mathbb{R}^+ \cup \{\infty\}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = (az + b)(cz + d)^{-1}$$

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- Using Vahlen matrices, we can give a neat proof.

Sketch of Proof of Isomorphism

- $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \leftrightarrow \begin{pmatrix} x_1 + x_2 & x_3 + x_4 i + x_5 j \\ x_3 - x_4 i - x_5 j & x_1 - x_2 \end{pmatrix} = M \in \mathbb{R} \ltimes SL^{\ddagger}(2, H_{\mathbb{R}})$

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- Thus, we have a map $SL^{\ddagger}(2, H_{\mathbb{R}}) \to SO(4, 1)$.

Sketch of Proof of Isomorphism, Part II

- Easy to check that
  - The kernel is $\{\pm 1\}$.
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- Therefore, $SL^{\ddagger}(2, H_{\mathbb{R}})/\{\pm 1\} \cong SO^0(4,1)$.
- Equivalently, $SL^{\ddagger}(2, H_{\mathbb{R}}) \cong \mathrm{Spin}(4,1)$.

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- Another standard example: $\mathrm{Mat}(2, \mathbb{R}) \cong \left(\frac{1,-1}{\mathbb{R}}\right)$.

Basic Facts about Quaternion Algebras

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  - Orthogonal involutions, which are all conjugate to one another—so, without loss of generality $(a + bi + cj + dij)^{\ddagger} = a + bi + cj - dij$.
- By an involution (of the first kind) I mean an F-linear map $\varphi : H \to H$ such that
  - $\varphi(\varphi(x)) = x$
  - $\varphi(xy) = \varphi(y)\varphi(x)$

Algebraic Groups

- For a field F an quaternion algebra H with orthogonal involution $\ddagger$, we define an algebraic group

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- By essentially the same argument as before, we have an isomorphism $SL^{\ddagger}(2,H)/\{\pm 1\} \cong SO^{0}(q)$, where

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- Conversely, for any indefinite, quinary quadratic form q, there is an H such that $SL^{\ddagger}(2,H)/\{\pm 1\} \cong SO^{0}(q)$.

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- One approach: take a subring $R \subset H$ that is also a lattice (i.e. it is an *order*), and consider $SL^{\ddagger}(2, R)$.
- This works perfectly well for $SL(2, \mathbb{C})$: take the ring of integers $\mathcal{O}$ of an imaginary quadratic field (e.g. $\mathbb{Z}[\sqrt{-2}]$), and define the corresponding Bianchi group $SL(2, \mathcal{O})$.

Orders Closed Under Involution

- For $SL^\ddagger(2, H)$, there is an obstruction

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  - Try to find maximal orders that are also $\ddagger$-orders. (Don't necessarily exist.)
  - Try to find $\ddagger$-orders that are not contained in other $\ddagger$-orders. (We'll call these *maximal $\ddagger$-orders*.)

Examples and non-Examples of Maximal ‡-Orders

Maximal orders:

$$\mathcal{O}_1 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+i+j}{2} \oplus \mathbb{Z}\frac{j+ij}{2} \subset \left(\frac{-1,-2}{\mathbb{Q}}\right)$$

$$\mathcal{O}_2 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{5+5i+3j-ij}{10} \subset \left(\frac{-1,-5}{\mathbb{Q}}\right)$$

$$\mathcal{O}_3 = \mathbb{Z} \oplus 9\mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{90+385i-63j+ij}{90} \subset \left(\frac{-1,-5}{\mathbb{Q}}\right)$$

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$$\mathcal{O}_2 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{5+5i+3j-ij}{10} \subset \left(\frac{-1,-5}{\mathbb{Q}}\right)$$
$$\mathcal{O}_3 = \mathbb{Z} \oplus 9\mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{90+385i-63j+ij}{90} \subset \left(\frac{-1,-5}{\mathbb{Q}}\right)$$

$$\mathcal{O}_1 \cap \mathcal{O}_1^{\ddagger} = \mathcal{O}_1$$
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  - $\mathrm{disc}(\ddagger) = \left\{ x^2 \middle| x \in H^- \right\} \in F^\times / F^{\times 2}$
- Can turn $\mathrm{disc}(\ddagger)$ into square-free ideal in F.

$$\iota : F^\times / F^{\times 2} \to \mathrm{Idl}(F)$$
$$[\lambda] \mapsto \bigcup_{\lambda \in [\lambda] \cap \mathfrak{o}} \lambda \mathfrak{o}$$

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- To get corresponding result for global result, apply localization.

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- S. preprint: Let $\mathcal{O}_1, \mathcal{O}_2$ be maximal ‡-orders over F. Then $\mathcal{O}_1 \cong \mathcal{O}_2$ if and only if $(\mathcal{O}_1)_\nu \cong (\mathcal{O}_2)_\nu$ for all places ν of F.

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- Corollary: the number of isomorphism classes over $\mathbb{Q}$ is either 1 or 2.

Comparison with Other Types of Orders

Maximal Orders	# of Orders	# of Isomorphism Classes
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QAs with involution		
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Moral of the story: maximal $\ddagger$-orders seem to be exceedingly nice, and criminally understudied.

Group Actions

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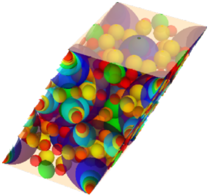
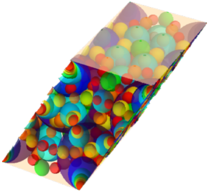
- Possible way to study $SL^{\ddagger}(2, H)$: fix a plane in $\mathbb{R}^{3}$, and consider the orbit under the action of $SL^{\ddagger}(2, H)$.

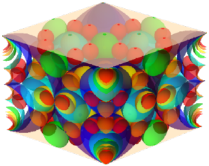
Sphere Packings



$$\left(\frac{-1,-6}{\mathbb{Q}}\right)$$



$$\left(\frac{-2,-5}{\mathbb{Q}}\right)$$



$$\left(\frac{-3,-1}{\mathbb{Q}}\right)$$

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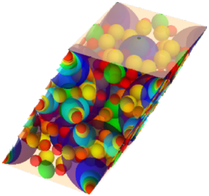

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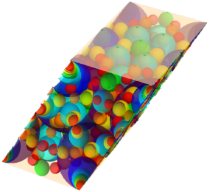

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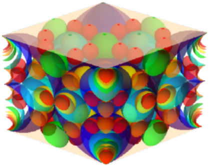

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- Can check that the bends (1/radius) are always integers.

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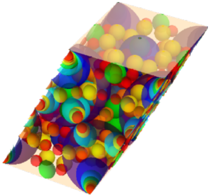

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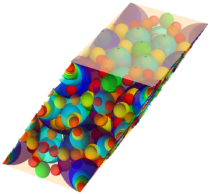

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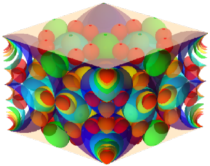

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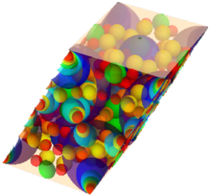

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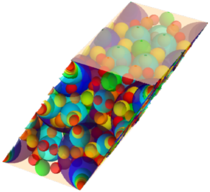

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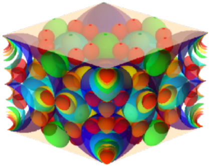

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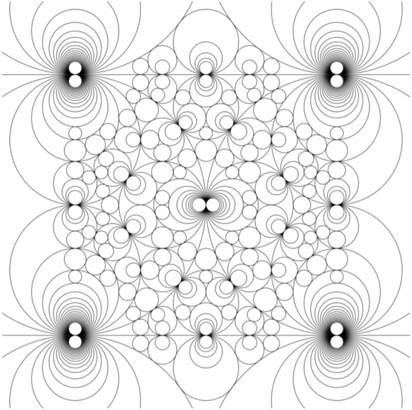
- Can therefore ask questions about which integers appear.
- This is easy: use strong approximation for algebraic groups.

Circle Packings



$SL(2, \mathbb{Z}[i])$

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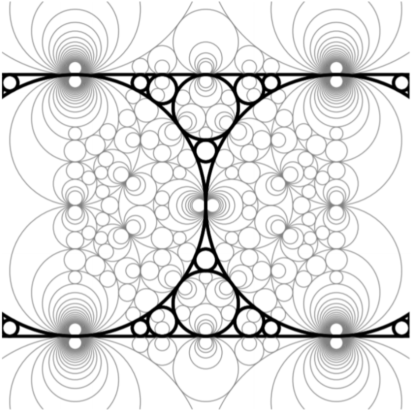

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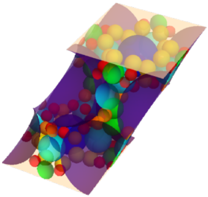
Apollonian-Like Packings



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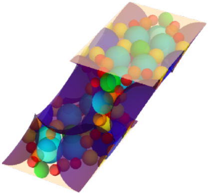

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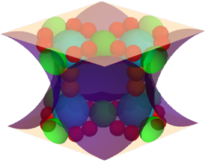

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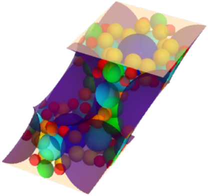

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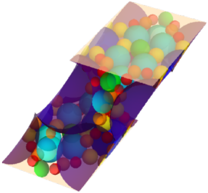

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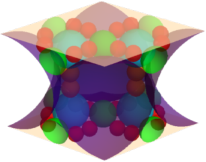

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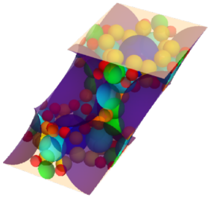

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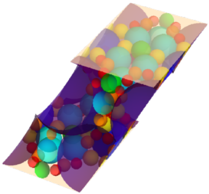

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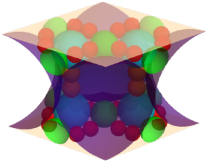

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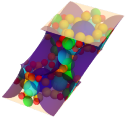
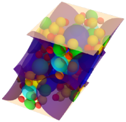
- S., thesis 2018: We can construct super-integral crystallographic packings from maximal ‡-orders. We can even give a partial classification.

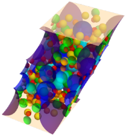
Super-Integral Crystallographic Sphere Packings



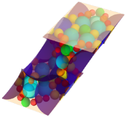
$\left(\frac{-1,-6}{\mathbb{Q}}\right)$



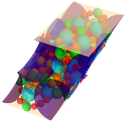
$\left(\frac{-1,-7}{\mathbb{Q}}\right)$



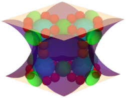
$\left(\frac{-1,-10}{\mathbb{Q}}\right)$



$\left(\frac{-2,-5}{\mathbb{Q}}\right)$



$\left(\frac{-2,-26}{\mathbb{Q}}\right)$



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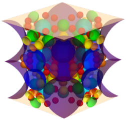

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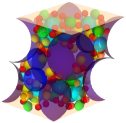

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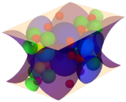
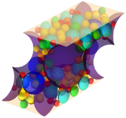


$\left(\frac{-7,-1}{\mathbb{Q}}\right)$



$\left(\frac{-11,-143}{\mathbb{Q}}\right)$

Future Work

- Groups $SL^{\ddagger}(2, \mathcal{O})$ are like Bianchi groups. How far does this analogy stretch?
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  - Easier for sphere packings. (Known results by Kontorovich, Nakamura, Dias.)