Connections between Integral Sphere Packings and Quaternion Algebras

Arseniy (Senia) Sheydvasser

April 16, 2018

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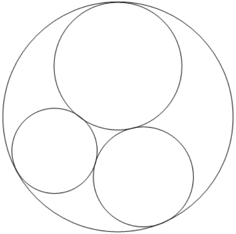
Apollonian Circle Packing



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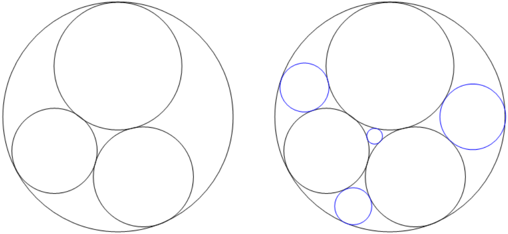

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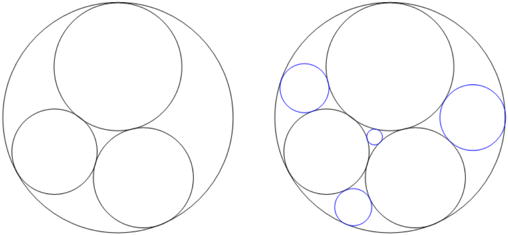

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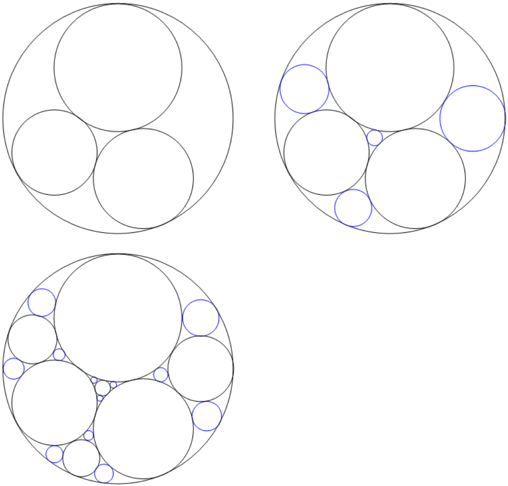

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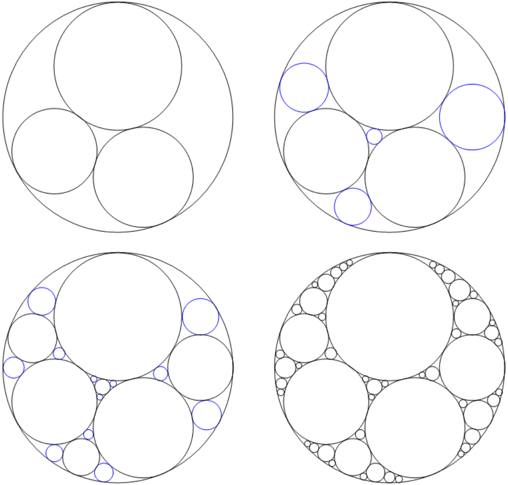

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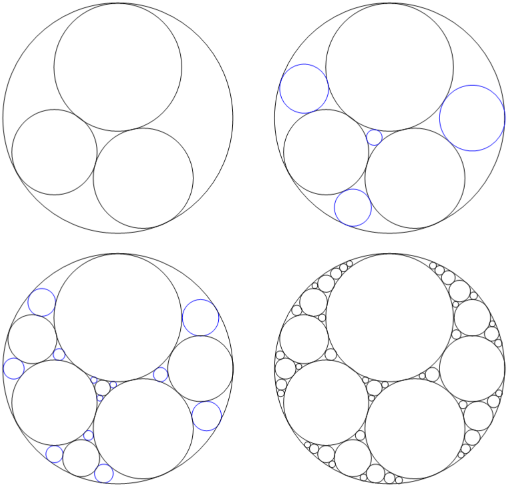

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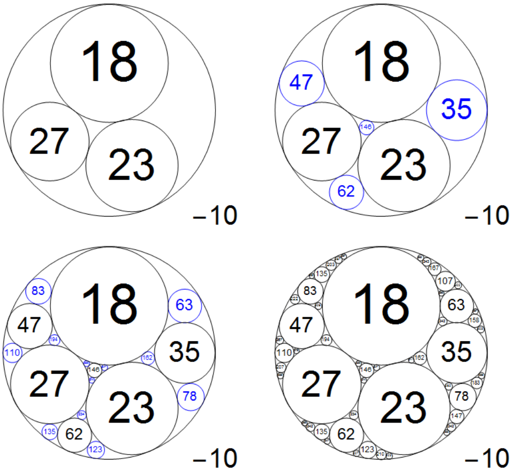

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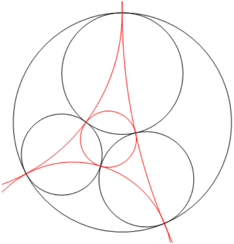

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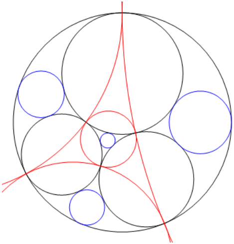

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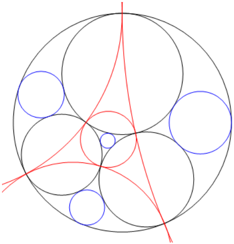

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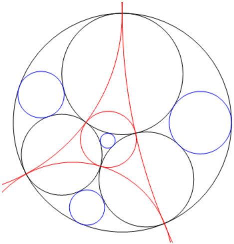

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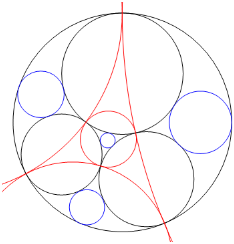

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$$\Gamma = \left\langle \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

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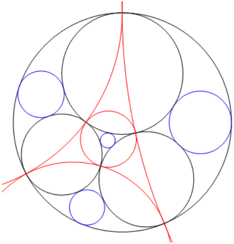

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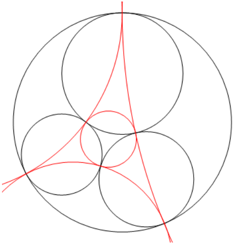
- Subgroup of $\overline{M\ddot{o}b(\mathbb{R}^2)}$.

Limit Set Interpretation



- Can extend the dual circles to spheres.

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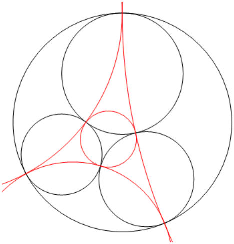

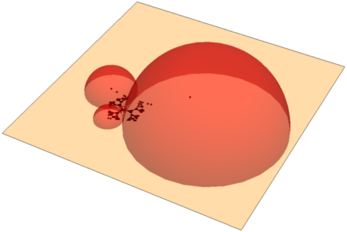

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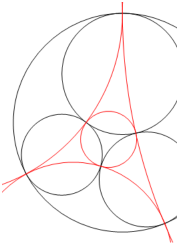

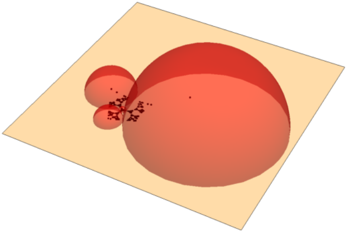

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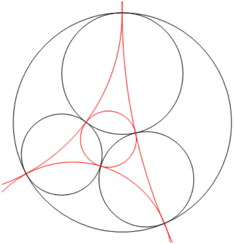
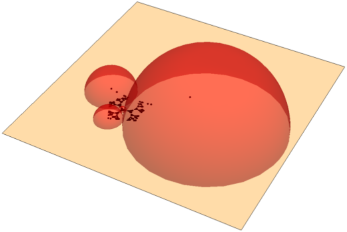
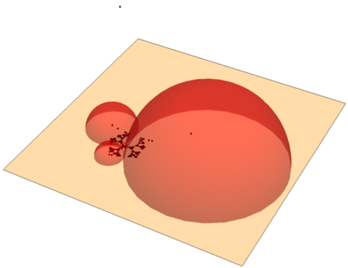




- Can extend the dual circles to spheres.
- Group generators are reflections in hyperbolic upper half-space.
- The closure of the Apollonian circle packing is the limit set of Γ.

Properties of Γ



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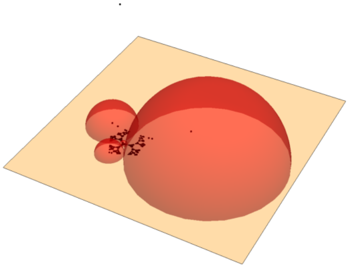

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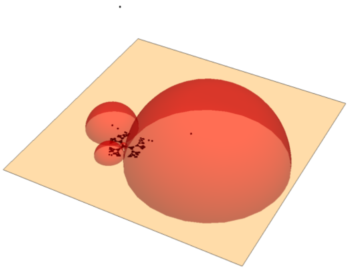

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- Γ is *thin*.

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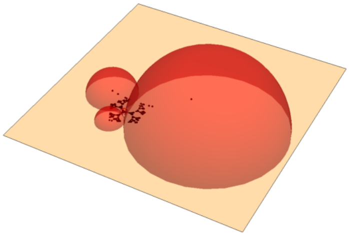

- Γ is a hyperbolic reflection group.
  - That is, the fundamental domain of Γ has only finitely many sides, and Γ is generated by reflections through those sides.
- Γ is *thin*.
  - That is, it is discrete, Zariski-dense, and the fundamental domain has infinite volume.

General Setting

Definition

An *n-sphere packing* is a collection of at least 3 oriented n-spheres such that their interiors do not intersect, and their union (with interiors) is dense in $\mathbb{R}^{n+1} \cup \{\infty\}$.

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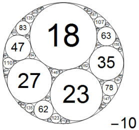
We say a crystallographic packing is *integral* if all of the bends (1/radius) are integers (possibly after scaling by some constant $C > 0$).

Some Known Constructions for $n = 1$

Apollonian gasket



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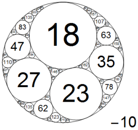
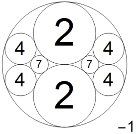


Octahedral packing (Guettler, Mallows 2010)



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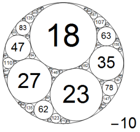

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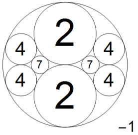
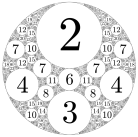


Orbits of $\mathbb{R}$ under action of $SL(2, \mathcal{O}_K)$ (Stange 2014, 2015)



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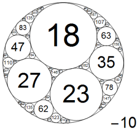

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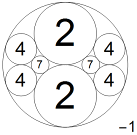

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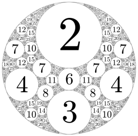
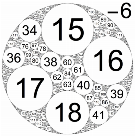


Geometrizing polyhedra (Kontorovich, Nakamura 2017)



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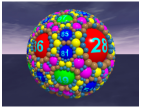

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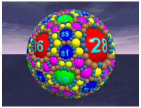

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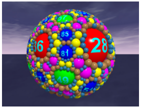

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Integral Boyd-Maxwell packings (Dolgachev 2014)

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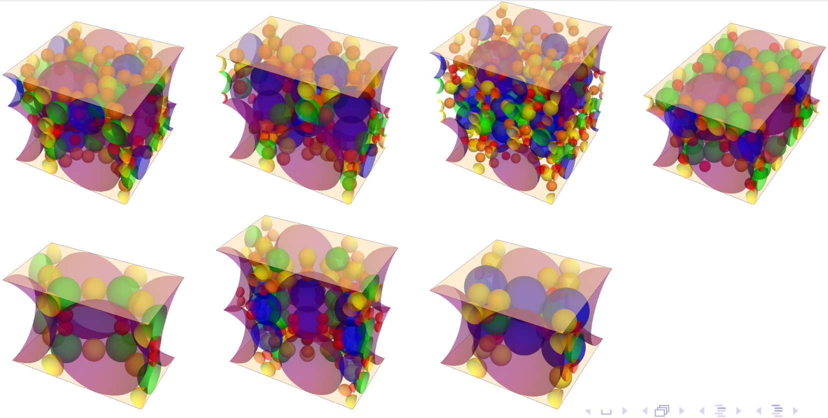

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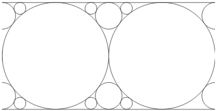
Example of Super Packing



- Start with the crystallographic packing.

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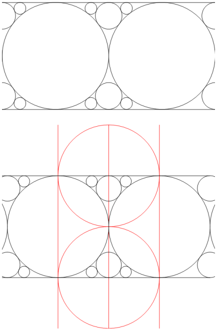

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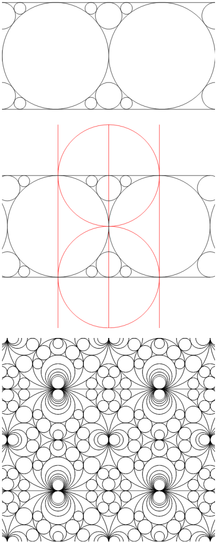

- Start with the crystallographic packing.
- Find hyperbolic reflections that map it back to itself.
- Look at orbit under super-group.

Super-Integral Crystallographic Packings

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Theorem (Kontorovich, Nakamura 2017)

If P is a super-integral crystallographic packing, then its super-group is arithmetic.

- So, what we want to do is study arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^3)$.

An Accidental Isomorphism

- A dimension down, this is easy to do due to the accidental isomorphisms

$$\begin{aligned} \mathsf{Isom}^0(\mathbb{H}^2) \cong SO^0(3,1) &\cong PSL(2,\mathbb{C}) \\ \mathsf{Spin}(3,1) &\cong SL(2,\mathbb{C}). \end{aligned}$$

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- We get this isomorphism due to the action of $SL(2,\mathbb{C})$ on $\mathbb{C}$ via Möbius transformations.
- In $\mathbb{R}^3$, we can again use the accidental isomorphism

$$\begin{aligned} \mathrm{Isom}^0(\mathbb{H}^3) \cong SO^0(4,1) &\cong PSL^{\ddagger}(2,H_{\mathbb{R}}) \\ \mathrm{Spin}(3,1) &\cong SL(2,H_{\mathbb{R}}). \end{aligned}$$

An Accidental Isomorphism

$$H_{\mathbb{R}} = \left(\frac{-1,-1}{\mathbb{R}}\right)$$

$$(w + xi_{\mathbb{R}} + yj_{\mathbb{R}} + zi_{\mathbb{R}}j_{\mathbb{R}})^{\ddagger} = w + xi_{\mathbb{R}} + yj_{\mathbb{R}} - zi_{\mathbb{R}}j_{\mathbb{R}}$$

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$$SL^{\ddagger}(2, H_{\mathbb{R}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in H_{\mathbb{R}},\ ab^{\ddagger} \in H_{\mathbb{R}}{}^{+}, \right.$$

$$\left. cd^{\ddagger} \in H_{\mathbb{R}}{}^{+},\ ad^{\ddagger} - bc^{\ddagger} = 1 \right\}$$

$$PSL^{\ddagger}(2, H_{\mathbb{R}}) = SL^{\ddagger}(2, H_{\mathbb{R}})/\{\pm 1\}.$$

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- Given any definite, rational quaternion algebra H and an (orthogonal) involution $\ddagger$, the group $SL^{\ddagger}(2, H)$ acts on H^{+} by Möbius transformations.

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- This yields an isomorphism with $\mathrm{Spin}(4,1)$.
- We want to look at arithmetic subgroups of such groups.
- Specifically, fix a plane $S_j \subset H^+$, and look at its orbit under an arithmetic subgroup $\tilde{\Gamma}$—we shall want to study whether this is the super-packing of some super-integral crystallographic packing.

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Non-Example

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Maximal ‡-Orders

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Given a quaternion algebra H over a local or global field F, with an orthogonal involution ‡, we say an order $\mathcal{O}$ of H is a ‡-*order* if $\mathcal{O} = \mathcal{O}^\ddagger$.

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- Maximal ‡-orders are easy to classify over local fields. (Scharlau 1974) (S. 2018)
- We can mostly reduce to this case via localization.
- In particular, if we look at orders such that $\mathcal{O} \cap \mathbb{Q}(i)$ has class number 1 (we shall call such orders *Heegner orders*), then we can give a complete list.

Heegner Orders

Theorem (S. 2018)

Let $H = \left(\frac{-m,-n}{\mathbb{Q}}\right)$ and $\mathcal{O}$ a Heegner order. Then $\mathcal{O}$ is one of the orders given below.

m	$\mathcal{O}$	Conditions
$m=1$	$\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \frac{1+i+j+ij}{2}$	if $n \equiv 1 \mod 4$
	$\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+i+j}{2} \oplus \mathbb{Z}\frac{j+ij}{2}$	if $n \equiv 2 \mod 4$
	$\begin{cases} \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+j}{2} \oplus \mathbb{Z}\frac{i+ij}{2} \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i+j}{2} \oplus \mathbb{Z}\frac{1+ij}{2} \end{cases}$	if $n \equiv -1 \mod 4$

Heegner Orders

$$m = 2 \left| \begin{array}{ll} \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+i+j}{2} \oplus \mathbb{Z}\frac{i+ij}{2} & \text{if } n \equiv 1 \mod 4 \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+j}{2} \oplus \mathbb{Z}\frac{i+ij}{2} & \text{if } n \equiv -1 \mod 4 \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i+j}{2} \oplus \mathbb{Z}\frac{2+2j+ij}{4} & \text{if } n/2 \equiv 1 \mod 8 \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i+j}{2} \oplus \mathbb{Z}\frac{2+ij}{4} & \text{if } n/2 \equiv 3 \mod 8 \\ \begin{cases} \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{2+i+j}{4} \oplus \mathbb{Z}\frac{i-j+ij}{4} \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{2+i-j}{4} \oplus \mathbb{Z}\frac{i+j+ij}{4} \end{cases} & \text{if } n/2 \equiv -3 \mod 8 \\ \begin{cases} \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i+j}{4} \oplus \mathbb{Z}\frac{2+ij}{4} \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i-j}{4} \oplus \mathbb{Z}\frac{2+ij}{4} \end{cases} & \text{if } n/2 \equiv -1 \mod 8 \end{array} \right.$$

Heegner Orders

$m = 7$	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{j+ij}{2}$	if $7 \nmid n$
	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{7j+ij}{14}$	if $7 \mid n$
$m \equiv 3 \mod 8$	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{j+ij}{2}$	if $m \nmid n$
	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{mj+ij}{2m}$	if $m \mid n$ and $\left(\frac{n/m}{m}\right) = 1$
	$\begin{cases} \mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}\frac{ti+j}{m} \oplus \mathbb{Z}\frac{mj+ij}{2m} \\ \mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}\frac{ti-j}{m} \oplus \mathbb{Z}\frac{mj+ij}{2m} \end{cases}$	if $m \mid n$ and $\left(\frac{n/m}{m}\right) = -1$

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(Here t is chosen such that $t^2 + n/m \equiv 0 \mod m$.)

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If $\mathcal{O}$ is a maximal $\ddagger$-order such that $\mathcal{O} \cap \mathbb{Q}(i)$ has class number 1, we call it a *Heegner order*.

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Definition

If $\mathcal{O}$ is a maximal $\ddagger$-order such that $\mathcal{O} \cap \mathbb{Q}(i)$ has class number 1, we call it a *Heegner order*. If, furthermore, $\mathcal{O}$ is invariant under the map

$$w + xi + yj + zij \mapsto w + xi - yj + zij,$$

we call it a *symmetric Heegner order*.

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- Let $\mathcal{O}$ be a symmetric Heegner order. Let $\mathcal{S}_{\mathcal{O},j}$ be the orbit of the plane S_j under the action of $SL^{\ddagger}(2, \mathcal{O})$. Then

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- Furthermore, let $E^{\ddagger}(2, \mathcal{O})$ be the smallest group containing $SL\left(2, \mathcal{O} \cap \mathbb{Q}(i)\right)$ and

$$\begin{pmatrix} u & 0 \\ 0 & {u^{\ddagger}}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

for all $u \in \mathcal{O}^{\times}$, $h \in \mathcal{O} \cap H^{+}$.

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- Then there is a bijection

$$SL^{\ddagger}(2,\mathcal{O})/E^{\ddagger}(2,\mathcal{O}) \to \{\text{tangency-connected components of } \mathcal{S}_{\mathcal{O},j}\}$$
$$\gamma E^{\ddagger}(2,\mathcal{O}) \mapsto \gamma E^{\ddagger}(2,\mathcal{O})\hat{S}_j$$

Candidates for Superpackings

- Putting this together, we can enumerate all sets $\mathcal{S}_{\mathcal{O},j}$ that are candidates for being (tangency-connected) superpackings.

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Theorem (S. 2018)

The sphere collection $\mathcal{S}_{\mathcal{O},j}$ is integral, tangential, dense, and tangency-connected for only a finite number of isomorphism classes of $H, \mathcal{O}$, given below.

H	$\mathcal{O}$
$\left(\frac{-1,-6}{\mathbb{Q}}\right)$	$\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+i+j}{2} \oplus \mathbb{Z}\frac{j+ij}{2}$
$\left(\frac{-1,-7}{\mathbb{Q}}\right)$	$\begin{cases} \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+j}{2} \oplus \mathbb{Z}\frac{i+ij}{2} \\ \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{i+j}{2} \oplus \mathbb{Z}\frac{1+ij}{2} \end{cases}$
$\left(\frac{-1,-10}{\mathbb{Q}}\right)$	$\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+i+j}{2} \oplus \mathbb{Z}\frac{j+ij}{2}$
$\left(\frac{-2,-5}{\mathbb{Q}}\right)$	$\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\frac{1+i+j}{2} \oplus \mathbb{Z}\frac{i+ij}{2}$
$\left(\frac{-3,-1}{\mathbb{Q}}\right)$	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{j+ij}{2}$
$\left(\frac{-3,-2}{\mathbb{Q}}\right)$	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{j+ij}{2}$
$\left(\frac{-7,-1}{\mathbb{Q}}\right)$	$\mathbb{Z} \oplus \mathbb{Z}\frac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{j+ij}{2}$.

The two sphere collections over $\left(\frac{-1,-7}{\mathbb{Q}}\right)$ are conformally equivalent—all other collections are conformally inequivalent.

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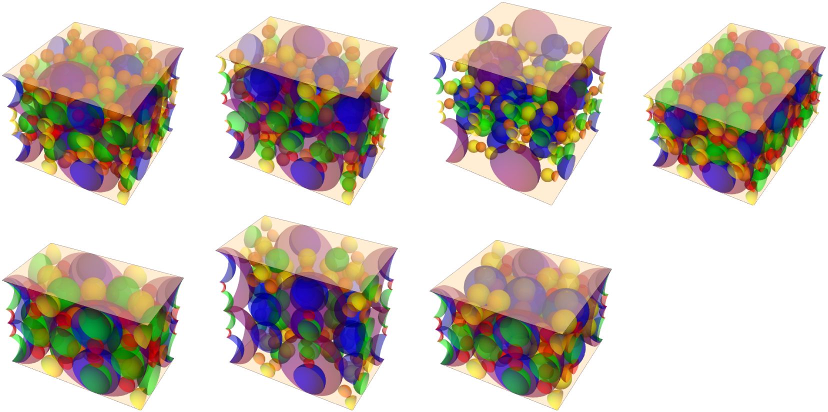

Integral Crystallographic Sphere Packings

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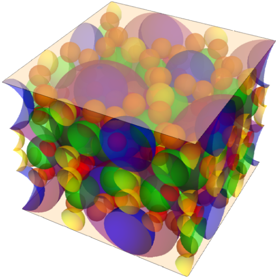
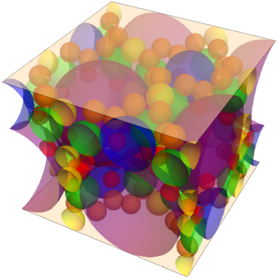


$\Rightarrow$



Integral Crystallographic Sphere Packings

Theorem (S. 2018)

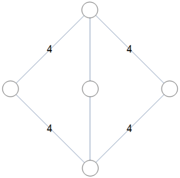
The sphere collections $\mathcal{S}_{\mathcal{O},j}$ corresponding to maximal ‡-orders

$$\begin{aligned}
\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\tfrac{1+i+j}{2} \oplus \mathbb{Z}\tfrac{j+ij}{2} \quad &\subset \left(\frac{-1,-6}{\mathbb{Q}}\right) \\
\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\tfrac{1+j}{2} \oplus \mathbb{Z}\tfrac{i+ij}{2} \quad &\subset \left(\frac{-1,-7}{\mathbb{Q}}\right) \\
\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\tfrac{1+i+j}{2} \oplus \mathbb{Z}\tfrac{j+ij}{2} \quad &\subset \left(\frac{-1,-10}{\mathbb{Q}}\right) \\
\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\tfrac{1+i+j}{2} \oplus \mathbb{Z}\tfrac{i+ij}{2} \quad &\subset \left(\frac{-2,-5}{\mathbb{Q}}\right) \\
\mathbb{Z} \oplus \mathbb{Z}\tfrac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\tfrac{j+ij}{2} \quad &\subset \left(\frac{-3,-1}{\mathbb{Q}}\right) \\
\mathbb{Z} \oplus \mathbb{Z}\tfrac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\tfrac{j+ij}{2} \quad &\subset \left(\frac{-3,-2}{\mathbb{Q}}\right) \\
\mathbb{Z} \oplus \mathbb{Z}\tfrac{1+i}{2} \oplus \mathbb{Z}j \oplus \mathbb{Z}\tfrac{j+ij}{2} \quad &\subset \left(\frac{-7,-1}{\mathbb{Q}}\right)
\end{aligned}$$

are the superpackings of super-integral crystallographic packings $\mathcal{P}_{\mathcal{O},j}$.

Integral Crystallographic Sphere Packings

- The packing corresponding to $\left(\frac{-1,-6}{\mathbb{Q}}\right)$ is the Maxwell-Boyd packing corresponding to the Coxeter diagram



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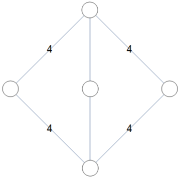

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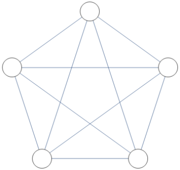

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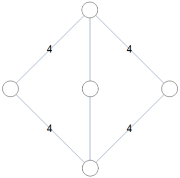

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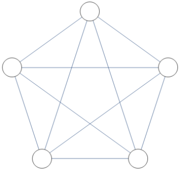

- All other packings are non-conformal to any known (to me) integral crystallographic packings.